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PIVOTAL VOTERS
A New Proof of Arrow's Theorem

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This paper presents a new proof of Arrow's 'General Possibility Theorem', focusing on the way how 'social' preferences change in response to changes in the preferences of individuals, under given social welfare functions. One individual is *pivotal* for a pair of alternatives at a preference profile if he can change the social ordering of these alternatives by just changing his preferences. Our proof of Arrow's theorem consists in showing that, under Paretian and independent of irrelevant alternatives social welfare functions, pivotality must be concentrated in the hands of a single individual— the dictator.

It is hoped that this alternative way of looking at a classical problem might be complementary with others in reaching a better understanding of how to design mechanisms for collective decision-making. In particular, the idea of pivotality has proved basic in the literature on strategy-proof allocation of public goods, and it may be a useful bridge between this parallel flow of work and that of social choice theory.

In order to emphasize the essentials, this version concentrates on the simple case where the number of alternatives is finite and both individual and social preferences are strict (no indifference allowed). The proof for the general case is similar, forthcoming and available upon request.

Let $I = \{1,2,\ldots,n\}$, an initial segment of the integers. Elements of I are called the *individuals*. Let A be a finite set. Elements of A are denoted by x, y, $z,\ldots$, and are called the *alternatives*. Let $\mathcal{P}$ be the set of complete, transitive, asymmetric binary relations on A. Elements of $\mathcal{P}$ are denoted by P, P', P_i, P_j, ... and are called *preference orderings*. We say that $x, y \in A$ *are contiguous* in $P \in \mathcal{P}$ iff $(\forall z \notin \langle x, y \rangle)[zPx \leftrightarrow zPy]$. Let $\mathcal{P}^n$ stand for the n-fold Cartesian product of $\mathcal{P}$. Elements of $\mathcal{P}^n$ are denoted by $\boldsymbol{P}$, $\boldsymbol{P}'$, ...and are called *preference profiles*.

When there is no ambiguity, P_i stands for the ith element of $\boldsymbol{P}$, P_j' for the jth element of $\boldsymbol{P}'$, etc. Given $\boldsymbol{P} \in \mathcal{P}^n$ and $P' \in \mathcal{P}$, $\boldsymbol{P}/_iP'$ denotes the profile $\boldsymbol{P}'$ where $P_i' = P'$ and $(\forall j \neq i) P_j = P_j'$.

A Social Welfare Function (SWF) is a function w: $\mathcal{P}^n \to \mathcal{P}$. Whenever there is no ambiguity, we denote $w(\boldsymbol{P})$ by P, $w(\boldsymbol{P}')$ by P', etc.

A SWF w is *Paretian* iff $(\forall \boldsymbol{P} \in \mathcal{P}^n)(\forall x, y \in A)[(\forall i \in I) xP_iy \to xPy]$.

A SWF w is *independent of irrelevant alternatives* (IIA) iff $(\forall \boldsymbol{P}, \boldsymbol{P}' \in \mathcal{P}^n)(\forall x, y \in A)[(\forall i \in I)(xP_iy \leftrightarrow xP_i'y) \to (xPy \leftrightarrow xP'y)]$. [1] A SWF w is *dictatorial* iff $\exists d \in I$ such that $(\forall x, y \in A)(\forall \boldsymbol{P} \in \mathcal{P}^n)[xP_dy \to xPy]$. Otherwise, w is *non-dictatorial*.

Individual $i \in I$ is *pivotal* at profile $\boldsymbol{P} \in \mathcal{P}^n$ under a social welfare function w iff $\exists P' \in \mathcal{P}$ such that $w(\boldsymbol{P}/_iP') \neq w(\boldsymbol{P})$. Individual $i \in I$ is *xy-pivotal* at $\boldsymbol{P} \in \mathcal{P}^n$ under SWF w iff $\exists P'$ such that $xw(\boldsymbol{P})y \leftrightarrow yw(\boldsymbol{P}/_iP')x$.

Thus, an individual is pivotal at profile $\boldsymbol{P}$ if he can change the 'social' preferences by just changing his preferences. It is xy-pivotal if he can change the social preferences in such a way that the relative position of x and y is reversed.

Remark that when w is IIA, if i is xy-pivotal at $\boldsymbol{P}$, it is also xy-pivotal at any $\boldsymbol{P}'$ such that $(\forall j \neq i, j \in I)[xP_jy \leftrightarrow xP_j'y]$. For IIA social welfare functions, we say that individual $i \in I$ is *positively xy-pivotal* at $\boldsymbol{P} \in \mathcal{P}^n$ under SWF w iff it is xy-pivotal at $\boldsymbol{P}$ and $xw(\boldsymbol{P}/_iP')y$ for all $P' \in \mathcal{P}$ such that $xP_i'y$.

Arrow's Theorem. Let $\#A > 2$. There exists no social welfare function which is Paretian, independent of irrelevant alternatives and non-dictatorial.

Proof. We start from any *given* Paretian and IIA social welfare function w, and prove that some individual $d \in I$ will be positively pivotal under w for any ordered pair of alternatives at every possible profile—i.e., a dictator. Our first step does not use the Pareto condition.

(1) *Under an IIA social welfare function, there can be no strong preference profile at which two individuals are pivotal for two different non-disjoint*

[1] The condition above is sometimes called binarity. It is *equivalent* in our context to Arrow's original condition of independence of irrelevant alternatives.

pairs of alternatives. For, let w be an IIA social welfare function, and suppose there were such a strong preference profile, individuals and alternatives. Then, without loss of generality, $\exists \boldsymbol{P} \in \mathcal{P}^n$, $P_i', P_j'' \in \mathcal{P}, i, j \in I$ and $x, y, z \in A$ such that [where $P' = w(\boldsymbol{P}/_i P_i'), P'' = w(\boldsymbol{P}/_j P_j'')$]:

(a) $xPyPz$, $yP'x$ and $zP''y$,

(b) x and y (*respectively* y *and* z) are contiguous in P_i and P_i' (*respectively in* P_j *and* P_j''),

(c) $sP_it \leftrightarrow sP_i't$ (respectively $sP_jt \leftrightarrow sP_j''t$) for all pairs of alternatives $\langle s,t\rangle \neq \langle x,y\rangle$ (respectively $\langle s,t\rangle \neq \langle y,z\rangle$).

(1) But then, i and j's pivotal would imply that, where $\hat{P} = w[(\boldsymbol{P}/_j P_j'')/_i P_i']$, $z\hat{P}y$, $y\hat{P}x$ and yet $x\hat{P}z$, a contradiction to the transitivity of $\hat{P}$.

(2) Since w is Paretian, *for every ordered pair of alternatives x, y there will be at least one preference profile at which some individual is positively xy-pivotal*. Select one such profile for each ordered pair of alternatives x, y, and denote it by $\boldsymbol{P}^{xy}$. Let $p(xy)$ be one specific individual which is pivotal at $\boldsymbol{P}^{xy}$.

(3) For any $x,y,z,w \in A$, and irrespectively of our choice of $\boldsymbol{P}^{xy}$ and $\boldsymbol{P}^{zw}$, it must be that $p(xy) = p(zw)$. That is, *for all profiles where some individual d is positively pivotal, this individual must be one and the same*. To prove it, suppose not —i.e., that for some possible choice of $\boldsymbol{P}^{xy}, \boldsymbol{P}^{zw}, p(xy)$ and $p(zw)$, we had $p(xy) \neq p(zw)$. Consider a profile $\hat{\boldsymbol{P}}$ where $(\forall i)[(x\hat{P}_iy \leftrightarrow xP_i^{xy}y)$ and $(x\hat{P}_iw \leftrightarrow zP_i^{zw}w)]$. Then

(a) If the pairs $(xy),(zw)$ have exactly one element in common, $p(xy) \neq p(zw)$ would contradict the Lemma, since both individuals are pivotal at $\hat{\boldsymbol{P}}$.

(b) If $x = w$ and $y = z$, it is impossible that, for any other $v \in A$, both $p(xy) = p(yv)$ and $p(yx) = p(xy)$ as required to avoid the contradiction found in (a).

(c) If $(xy),(zw)$ have no element in common, $p(xy) \neq p(zw)$ implies that either $p(xy) \neq p(yz)$ or $p(yz) \neq p(zw)$, leading again to the contradiction discussed in (a).

(4) *Individual d, as defined in 3, is a dictator, since it is positively pivotal for all pairs of alternatives at all profiles.* To prove it, suppose not —i.e., that for some $\boldsymbol{P} \in \mathcal{P}^n$ and some $x, y \in A$, d is not positively xy-pivotal at $\boldsymbol{P}$. Since, by (1), there are profiles where d is positively xy-pivotal, there must exist two profiles which only differ in the

preferences of one individual $h \neq d$, and such that d is positively xy-pivotal in one of these profiles while not in the other. But then, there must be a strong profile where $h \neq d$ is xy-pivotal, and thus one where h is xy-pivotal and d is yz-pivotal. This contradicts the Lemma.

Reference

Arrow, Kenneth J., 1963, Social choice and individual values, 2nd ed. (Wiley, New York).